Generating Functions for the Polynomials in d–Dimensional Semiclassical Wave Packets

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Abstract

We present a simple formula for the generating function for the polynomials in the d–dimensional semiclassical wave packets.

1 Introduction

The generating function for 1–dimensional semiclassical wave packets is presented in formula (2.47) of [2]. In this paper, we present and prove the d–dimensional analog.

This result has also been proven from a different point of view by Helge Dietert, Johannes Keller, and Stephanie Troppmann. See Lemma 3 and Section 3 (particularly Proposition 16) of [1]. We have also received a conjecture from Tomoki Ohsawa [3] that this result could be proved abstractly by using the formula for products of Hermite polynomials and the action of the metaplectic group.

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The semiclassical wave packets depend on two invertible $d\times d$ complex matrices A and B that are always assumed to satisfy

$$A^*\,B+B^*\,A\;=\;2\,I\qquad\text{and}\qquad A^t\,B-B^t\,A\;=\;0.$$

They also depend on a phase space point $(a,\,\eta)$ that plays no role in the present work. After chosing a branch of the square root, we define

$$\begin{aligned}\varphi_0(A,\,B,\,\hbar,\,a,\,\eta,\,x)\;&=\;\pi^{-1/4}\,\hbar^{-1/4}\,(\det\,A)^{-1/2}\\&\qquad\times\quad\exp\left(-\,\frac{\langle(x-a),\,B\,A^{-1}\,(x-a)\rangle}{2\,\hbar}\;+\;i\,\frac{\langle\eta,\,(x-a)\rangle}{\hbar}\right).\end{aligned}$$

Here, and for the rest of this paper, we regard $\mathbb{R}^d$ as being embedded in $\mathbb{C}^d$, and for any two vectors $a\in\mathbb{C}^d$ and $b\in\mathbb{C}^d$, we use the notation

$$\langle\,a,\,b\,\rangle\;=\;\sum_{j=1}^d\,\overline{a_j}\,b_j.$$

For $1\le l\le d$, we define the l^{th} raising operator

$$\mathcal{R}_l\;=\;\mathcal{A}_l(A,\,B,\,\hbar,\,0,\,0)^*\;=\;\frac{1}{\sqrt{2\,\hbar}}\,(\langle\,B\,e_l,\,(x-a)\,\rangle\;-\;i\,\langle\,A\,e_l,\,(-\,i\,\hbar\nabla-\eta)\,\rangle)\,.$$

Then recursively, for any multi-index k, we define

$$\varphi_{k+e_l}(A,\,B,\,\hbar,\,a,\,\eta,\,x)\;=\;\frac{1}{\sqrt{k_l+1}}\,\mathcal{R}_l(\varphi_k(A,\,B,\,\hbar,\,a,\,\eta))(x).$$

For fixed A, B, $\hbar$, a, η, these wave packets form an orthonormal basis indexed by k. It is easy to see that

$$\varphi_k(A,\,B,\,\hbar,\,a,\,\eta,\,x)\;=\;2^{-|k|/2}\,(k!)^{-1/2}\,P_k(A,\,\hbar,\,(x-a))\,\varphi_0(A,\,B,\,\hbar,\,a,\,\eta,\,x),$$

where $P_k(A,\,\hbar,\,(x-a))$ is a polynomial of degree $|k|$ in $(x-a)$, although from this definition, it is not immediately obvious that $P_k(A,\,\hbar,\,(x-a))$ is independent of B.

Since they play no interesting role in what we are doing here, we henceforth assume $a=0$ and $\eta=0$.

Our main result is the following:

Theorem 1.1 *The generating function for the family of polynomials* $P_k(A,\,\hbar,\,x)$ *is*

$$G(x,\,z)\;=\;\exp\left(-\left\langle\,\overline{z},\,A^{-1}\,\overline{A}\,z\right\rangle+\frac{2}{\sqrt{\hbar}}\left\langle\,\overline{z},\,A^{-1}\,x\right\rangle\right).$$

I.e.,

$$G(x,\,z)\;=\;\sum_k\;P_k(A,\,\hbar,\,x)\;\frac{z^k}{k!}.$$

Remark We make the unconventional definition $|A|=\sqrt{A\,A^*}$. By our conditions on the matrices A and B, this forces $|A|$ to be real symmetric and strictly positive. We also have the polar decomposition $A=|A|\,U_A$, where U_A is unitary. With this notation, we can write

$$G(x,\,z)\;=\;\exp\left(-\left\langle\,U_A\,\overline{z},\,\overline{U_A}\,z\right\rangle+\frac{2}{\sqrt{\hbar}}\left\langle\,U_A\,\overline{z},\,|A|^{-1}\,x\right\rangle\right).$$

This equivalent formula is the one we shall actually prove.

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2 Proof of the Theorem

We begin with a lemma that provides an alternative formula for $\mathcal{R}_l$. From this formula and an induction on $|k|$, one can easily prove that $P_k(A,\,\hbar,\,x)$ is independent of B, because

$$\overline{\varphi_0(A,\,B,\,\hbar,\,0,\,0,\,x)}\;\varphi_0(A,\,B,\,\hbar,\,0,\,0,\,x)\;=\;\pi^{-1/2}\,\hbar^{-1/2}\,|\det A|^{-1}\,\exp\left(-\,\frac{\langle x,\,|A|^{-2}\,x\rangle}{\hbar}\right).$$

Lemma 2.1 *For any* $\psi\in\mathcal{S}$,

$$(R_l\,\psi)(x)\;=\;-\sqrt{\frac{\hbar}{2}}\;\frac{1}{\overline{\varphi_0(A,\,B,\,\hbar,\,0,\,0,\,x)}}\;\left\langle\,A\,e_l,\,\nabla\left(\,\overline{\varphi_0(A,\,B,\,\hbar,\,0,\,0,\,x)}\;\psi(x)\right)\right\rangle.$$

Proof: The gradient on the right hand side of the equation in the lemma can act either on the $\overline{\varphi_0}$ or on the ψ. So, we get two terms when we compute this:

$$\sqrt{\frac{\hbar}{2}}\left(\frac{1}{2\,\hbar}\sum_{j=1}^{d}\left\langle\,A\,e_l,\;\left(e_j\left(\langle\,e_j,\,\overline{B}\,\overline{A}^{\;-1}\,x\rangle+\langle\,x,\,\overline{B}\,\overline{A}^{\;-1}\,e_j\rangle\right)\right)\right\rangle\,\psi(x)\right.$$

$$\left.-\quad\langle\,A\,e_l,\;(\nabla\,\psi)(x)\,\rangle\right).$$

The second term here is precisely the second term $\dfrac{1}{\sqrt{2\,\hbar}}\,(-\,i\,\langle\,A\,e_l,\,(-\,i\,\hbar\nabla)\,\psi(x)\,\rangle)$, in the expression for $(R_l\psi)(x)$. So, we need only show the first term here equals the first term, $\dfrac{1}{\sqrt{2\,\hbar}}\,\langle\,B\,e_l,\,x\,\rangle\,\psi(x)$, in the expression for $(R_l\psi)(x)$.

To do this, we begin by noting that the first term here equals

$$
\begin{aligned}
& \frac{1}{2\sqrt{2\,\hbar}}\,\sum_{j=1}^{d}\,\Big\langle\,A\,e_l,\;\Big(e_j\,\Big(\langle\,e_j,\,\overline{B}\,\overline{A}^{\,-1}\,x\rangle\,+\,\langle\,x,\,\overline{B}\,\overline{A}^{\,-1}\,e_j\rangle\Big)\Big\rangle\,\psi(x) \\
=\;\; & \frac{1}{2\sqrt{2\,\hbar}}\,\sum_{j=1}^{d}\,\Big\langle\,A\,e_l,\;\Big(e_j\,\Big(\langle\,e_j,\,\overline{B}\,\overline{A}^{\,-1}\,x\rangle\,+\,\overline{\langle\,\overline{B}\,\overline{A}^{\,-1}\,e_j,\,x\,\rangle}\Big)\Big\rangle\,\psi(x) \\
=\;\; & \frac{1}{2\sqrt{2\,\hbar}}\,\sum_{j=1}^{d}\,\Big\langle\,A\,e_l,\;\Big(e_j\,\Big(\langle\,e_j,\,\overline{B}\,\overline{A}^{\,-1}\,x\rangle\,+\,\langle\,B\,A^{-1}\,e_j,\,x\,\rangle\Big)\Big\rangle\,\psi(x) \\
=\;\; & \frac{1}{2\sqrt{2\,\hbar}}\,\sum_{j=1}^{d}\,\Big\langle\,A\,e_l,\;\Big(e_j\,\Big(\langle\,e_j,\,\overline{B}\,\overline{A}^{\,-1}\,x\rangle\,+\,\langle\,e_j,\,\left(A^{-1}\right)^{*}\,B^{*}\,x\,\rangle\Big)\Big\rangle\,\psi(x) \\
=\;\; & \frac{1}{\sqrt{2\,\hbar}}\,\left\langle\,A\,e_l,\;\frac{\overline{B}\,\overline{A}^{\,-1}\,+\,(A^{-1})^{*}\,B^{*}}{2}\,x\,\right\rangle\,\psi(x)
\end{aligned}
$$

Because of the relations satisfied by A and B, $B\,A^{-1}$ is (real symmetric) $+\,i$ (real symmetric). So, its conjugate, $\overline{B}\,\overline{A}^{\,-1}$ has this same form. Thus, $\overline{B}\,\overline{A}^{\,-1}$ equals its transpose, which is $\left(A^{-1}\right)^{*}\,B^{*}$. So, the quantity of interest here equals

$$
\begin{aligned}
& \frac{1}{\sqrt{2\,\hbar}}\,\langle\,A\,e_l,\;\left(A^{-1}\right)^{*}\,B^{*}\,x\,\rangle\,\psi(x) \\
=\;\; & \frac{1}{\sqrt{2\,\hbar}}\,\langle\,e_l,\;A^{*}\,\left(A^{-1}\right)^{*}\,B^{*}\,x\,\rangle\,\psi(x) \\
=\;\; & \frac{1}{\sqrt{2\,\hbar}}\,\langle\,e_l,\;B^{*}\,x\,\rangle\,\psi(x) \\
=\;\; & \frac{1}{\sqrt{2\,\hbar}}\,\langle\,B\,e_l,\,x\,\rangle\,\psi(x),
\end{aligned}
$$

which is what we had to show. ∎

Proof of the Theorem: We prove the theorem by an induction on $|k|$. For $k=0$, the result is trivial since $P_0(A,\,\hbar,\,x)=1$.

Without ever computing an explicit formula for the polynomial p_k (which may be complicated), we prove inductively that

$$P_k(A,\,\hbar,\,x) \;=\; p_k(|A|^{-1}\,x/\sqrt{\hbar})$$

and

$$\left(\frac{\partial}{\partial z}\right)^k G(x,\,z) \;=\; p_k(|A|^{-1}\,x/\sqrt{\hbar}-\overline{U_A}\,z)\,G(x,\,z).$$

The result then follows by setting $z=0$.

For the induction step, it is sufficient to do the following for an arbitrary positive integer $l\le d$:
Assuming we have already proved these for some k, we prove them for the multi–index $k+e_l$.

To do this, we begin by noting that

$$\varphi_k(A,\,B,\,\hbar,\,0,\,0,\,x) \;=\; \frac{1}{\sqrt{k!}}\,\mathcal{R}^k(\varphi_0(A,\,B,\,\hbar,\,0,\,0))(x).$$

Also,

$$\varphi_k(A,\,B,\,\hbar,\,0,\,0,\,x) \;=\; 2^{-|k|/2}\,(k!)^{-1/2}\,P_k(A,\,\hbar,\,x)\,\varphi_0(A,\,B,\,\hbar,\,0,\,0,\,x).$$

So,

$$\mathcal{R}^k(\varphi_0(A,\,B,\,\hbar,\,0,\,0))(x) \;=\; 2^{-|k|/2}\,P_k(A,\,\hbar,\,x)\;\varphi_0(A,\,B,\,\hbar,\,0,\,0,\,x).$$

Thus, when we apply the l^{th} raising operator, the polynomial $P_k(A,\,\hbar,\,x)$ gets changed to $\dfrac{1}{\sqrt{2}}\,P_{k+e_l}(A,\,\hbar,\,x)$.

Assuming the induction hypothesis, when we differentiate $\dfrac{\partial^k G}{\partial z^k}$ with respect to z_l, the z_l derivative can act on the $G(x,\,z)$ or it can act on the $p_k(|A|^{-1}\,x/\sqrt{\hbar}-U_A\,z)$. When it acts on the $G(x,\,z)$, we obtain

$$2\,\left\langle U_A\,e_l,\;\left(|A|^{-1}\,x/\sqrt{\hbar}-\overline{U_A}\,z\right)\right\rangle\;p_k(A,\,\hbar,\,x)\;G(x,\,z). \tag{2.1}$$

Note that this result depends on the following calculation, with $G(x,\,z)$ written with the polar decomposition of A:

$$
\begin{aligned}
\frac{\partial G}{\partial z_k}(x,\,z) &= \left(-\,\langle\, U_A\, e_l,\, \overline{U_A}\, z\,\rangle \,-\, \langle\, U_A\, \overline{z},\, \overline{U_A}\, e_l\,\rangle \,+\, \frac{2}{\sqrt{\hbar}}\,\langle\, U_A\, e_l,\, |A|^{-1}\, x\,\rangle \right)\; G(x,\,z)\\
&= 2\,\left\langle\, U_A\, e_l,\, \left(|A|^{-1}\, x/\sqrt{\hbar} - \overline{U_A}\, z \right) \right\rangle\; G(x,\,z).
\end{aligned}
$$

When the $\dfrac{\partial}{\partial z_l}$ acts on the polynomial, we get

$$
\begin{aligned}
& -\,\left\langle\, \overline{(\nabla p_k)(|A|^{-1}\, x/\sqrt{\hbar} - \overline{U_A}\, z)},\; \overline{U_A}\, e_l \right\rangle\; G(x,\,z)\\
=\; & -\,\left\langle\, U_A\, e_l,\; (\nabla p_k)(|A|^{-1}\, x/\sqrt{\hbar} - \overline{U_A}\, z) \right\rangle\; G(x,\,z).
\end{aligned} \tag{2.2}
$$

Recall that

$$
(R_l\,\psi)(x) \;=\; -\,\sqrt{\frac{\hbar}{2}}\;\frac{1}{\overline{\varphi_0(A,\,B,\,\hbar,\,0,\,0,\,x)}}\;\left\langle\, A\, e_l,\, \nabla\, \left(\overline{\varphi_0(A,\,B,\,\hbar,\,0,\,0,\,x)}\;\psi(x) \right) \right\rangle,
$$

and that from our induction hypothesis,

$$
\overline{\varphi_0(A,\,B,\,\hbar,\,0,\,0,\,x)}\;\varphi_k(A,\,B,\,\hbar,\,0,\,0,\,x) \;=\; 2^{-|k|/2}\,(k!)^{-1/2}\,p_k(A,\,\hbar,\,x)\,e^{-\frac{\langle\, x,\,|A|^{-2}\,x\,\rangle}{\hbar}}.
$$

The gradient in $\mathcal{R}_l$ can act on the exponential or the $p_k(A,\,\hbar,\,x)$. When it acts on the exponential, we get

$$
\begin{aligned}
& 2^{-|k|/2}\;(k!)^{-1/2}\;p_k(A,\,\hbar,\,x)\;\sqrt{\frac{2}{\hbar}}\;\langle\, A\, e_l,\, |A|^{-2}\, x\,\rangle\;\varphi_0(A,\,B,\,\hbar,\,0,\,0,\,x)\\
=\; & 2^{-(|k|+1)/2}\;\sqrt{k_l+1}\;((k+e_l)!)^{-1/2}\\
& \qquad\times\;\; 2\,\left\langle\, U_A\, e_l,\, |A|^{-1}\, x/\sqrt{\hbar} \right\rangle\; p_k(A,\,\hbar,\,x)\;\varphi_0(A,\,B,\,\hbar,\,0,\,0,\,x).
\end{aligned} \tag{2.3}
$$

When the gradient in $\mathcal{R}_l$ acts on the $p_k(A,\,\hbar,\,x)$, we get

$$
\begin{aligned}
& -\,\sqrt{\frac{\hbar}{2}}\;2^{-|k|/2}\;(k!)^{-1/2}\;\langle\, A\, e_l,\, \nabla_x p_k(A,\,\hbar,\,x)\,\rangle\;\varphi_0(A,\,B,\,\hbar,\,0,\,0,\,x)\\
=\; & -\,2^{-(|k|+1)/2}\,(k!)^{-1/2}\,\left\langle A\, e_l,\; \sum_{j=1}^{d}\langle\, e_j,\, (\nabla p_k)(A,\,\hbar,\,x)\,\rangle\,|A|^{-1}\, e_j \right\rangle\;\varphi_0(A,\,B,\,\hbar,\,0,\,0,\,x)
\end{aligned}
$$

$$= -\, 2^{-(|k|+1)/2}\, (k!)^{-1/2} \left\langle A\, e_l,\ |A|^{-1}\, (\nabla p_k)(A,\, \hbar,\, x) \right\rangle \varphi_0(A,\, B,\, \hbar,\, 0,\, 0,\, x)$$

$$= -\, 2^{-(|k|+1)/2}\, \sqrt{k_l+1}\, ((k+e_l)!)^{-1/2}$$

$$\times \quad \langle U_A\, e_l,\ (\nabla p_k)(A,\, \hbar,\, x) \rangle\, \varphi_0(A,\, B,\, \hbar,\, 0,\, 0,\, x). \tag{2.4}$$

From (2.1) and (2.2) with $z = 0$, we obtain

$$2 \left\langle U_A\, e_l,\ |A|^{-1}\, x/\sqrt{\hbar} \right\rangle p_k(A,\, \hbar,\, x) \,-\, \left\langle U_A\, e_l,\ (\nabla p_k)(|A|^{-1}\, x/\sqrt{\hbar}) \right\rangle.$$

From (2.3) and (2.4) and taking into account the factor of $\sqrt{k_l+1}$ in $\mathcal{R}_l(\varphi_k) = \sqrt{k_l+1}\, \varphi_{k+e_l}$, we obtain

$$\begin{aligned} & P_{k+e_l}(A,\, \hbar,\, x) \\ =\ & 2 \left\langle U_A\, e_l,\ |A|^{-1}\, x/\sqrt{\hbar} \right\rangle p_k(A,\, \hbar,\, x) \,-\, \left\langle U_A\, e_l,\ (\nabla p_k)(|A|^{-1}\, x/\sqrt{\hbar}) \right\rangle. \end{aligned}$$

The quantities of interest contain the same polynomial evaluated at the appropriate arguments, and $P_{k+e_l}(A,\, \hbar,\, x) = p_{k+e_l}(A,\, \hbar,\, x)$. Since l is arbitrary, with $1 \le l \le d$, the result is true for all multi-indices with order $|k|+1$, and the induction can proceed. ■

References

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